

USACH-FM-00-02

Bosonized supersymmetric quantum mechanics and supersymmetry of parabosons (parafermions)*

Mikhail S. Plyushchay

*Departamento de Física, Universidad de Santiago de Chile,
Casilla 307, Santiago 2, Chile*

and

*Institute for High Energy Physics, Protvino, Russia
E-mail: mplyushc@lauca.usach.cl*

Abstract

We review the construction of minimally bosonized supersymmetric quantum mechanics and its relation to hidden supersymmetries in pure parabosonic (parafermionic) systems.

The simplest $N = 1$ supersymmetric quantum mechanical system is the superoscillator [1] given by the Hamiltonian

$$H = \frac{1}{2}\{b^+, b^-\} + \frac{1}{2}[f^+, f^-]. \quad (1)$$

In addition to H , the system is characterized by the two conserved integrals of motion (supercharges)

$$Q_{\pm} = b^{\mp} f^{\pm}, \quad (2)$$

which are quadratic in bosonic (b^{\pm}) and fermionic (f^{\pm}) creation-annihilation operators. The operator $\Gamma = [f^+, f^-]$ has a sense of the grading operator,

* Talk given at the International Workshop “Supersymmetry and Quantum Symmetries”, JINR, Dubna, Russia, July 26-31, 1999

$\Gamma^2 = 1$, which classifies all the operators into even (bosonic, B) and odd (fermionic, F) subsets according to the relations $[\Gamma, B] = 0$, $\{\Gamma, F\} = 0$. The even (H) and odd (Q_\pm) integrals of motion form the $N = 1$ superalgebra:

$$\{Q_+, Q_-\} = H, \quad Q_+^2 = Q_-^2 = 0, \quad [H, Q_\pm] = 0. \quad (3)$$

Realization of operators f^\pm in terms of Pauli matrices, $f^\pm = \sigma_\pm \equiv \frac{1}{2}(\sigma_1 \pm i\sigma_2)$, leads to the diagonal form for the grading operator, $\Gamma = \sigma_3$.

Simple change in (2) of bosonic operators b^\pm for the mutually hermitian conjugate linear differential operators

$$B^\pm = \frac{1}{\sqrt{2}} \left(W(x) \mp \frac{d}{dx} \right) \quad (4)$$

results in supersymmetric quantum mechanics (SUSYQM) [2, 3] generalizing the simplest $N = 1$ supersymmetric system (1), (2),

$$H = \frac{1}{2} \left(-\frac{d^2}{dx^2} + W^2(x) + W'(x)\sigma_3 \right), \quad Q_\pm = B^\mp \sigma_\pm, \quad (5)$$

with H and Q_\pm satisfying the same superalgebra (3).

The reflection operator R , $R^2 = 1$, $R\psi(x) = \psi(-x)$, introduces the \mathbb{Z}_2 -grading structure in the space of wave functions by classifying them as even and odd ones, $R\psi_\pm(x) = \pm\psi_\pm(x)$, $\psi_\pm = \frac{1}{2}(\psi(x) \pm \psi(-x))$. Since the \mathbb{Z}_2 -grading structure is the necessary element of SUSYQM, one can try to realize the latter without introducing independent fermionic operators or associated matrix structure but by declaring the reflection operator to be the grading operator, $\Gamma = R$. Indeed, first we note that the operator R gives a possibility to construct the operators $\Sigma_1 = \epsilon(x)$, $\Sigma_2 = i\epsilon(x)R$, $\Sigma_3 = R$ (with $\epsilon(x) = -1, 0, +1$ for $x < 0, = 0, > 0$), which satisfy the algebra of Pauli matrices: $\Sigma_i \Sigma_j = \delta_{ij} + i\epsilon_{ijk} \Sigma_k$. However, unlike the matrices σ_i , the operators Σ_i do not commute with x and d/dx . In particular, $\Sigma_3 = R$ anticommutes with these operators. The analogs of nilpotent operators σ_\pm take here the form $\Sigma_\pm \equiv \frac{1}{2}(\Sigma_1 \pm i\Sigma_2) = \epsilon(x)\Pi_\mp$, where $\Pi_\pm = \frac{1}{2}(1 \pm R)$ are the projectors, $\Pi_\pm^2 = \Pi_\pm$, $\Pi_+\Pi_- = 0$, $\Pi_+ + \Pi_- = 1$. Such a structure of operators Σ_\pm provides us with a hint for the construction of odd (anticommuting with the grading operator R) nilpotent supercharge operators by analogy with (5), (4),

$$Q_\pm = \frac{1}{\sqrt{2}} \left(W_\pm(x) \pm \frac{d}{dx} \right) \Pi_\pm, \quad (6)$$

where $W_-(x)$ is a superpotential to be, unlike the case of (5), an *odd function*. Defining the Hamiltonian as an anticommutator of Q_+ and Q_- , we get

$$H = \frac{1}{2} \left(-\frac{d^2}{dx^2} + W_-^2(x) - W'_-(x)R \right). \quad (7)$$

By the construction, it commutes with the grading operator R , i.e., is the even operator, and the operators (6), (7) form the $N = 1$ superalgebra (3).

The formally constructed *minimally* bosonized supersymmetric quantum mechanics [4, 5] given by Eqs. (6), (7) is related to the initial version of SUSYQM (5) in a nontrivial way. This can be observed immediately just by considering the simplest case of linear superpotential $W(x) = \epsilon x$, $\epsilon = +$ or $-$. Both these cases correspond to the same system of superoscillator (1), (2) with the spectrum $E = 0, 1, 1, 2, 2, \dots$, i.e., the supersymmetry in this case is exact. On the other hand, the bosonized version of SUSYQM with $W_-(x) = +x$ describes the system with exact supersymmetry ($E = 0, 2, 2, 4, 4, \dots$), whereas $W_-(x) = -x$ gives a system in the phase of spontaneously broken supersymmetry ($E = 1, 1, 3, 3, \dots$) [4, 5]. The specified spectra of these two different bosonized supersymmetric systems constitute together the spectrum of superoscillator (1). This observation helps to establish the exact relation between the two versions of SUSYQM: if the ordinary supersymmetric system is given by the superpotential being an odd function, $W(-x) = -W(x)$, then the bosonized version of SUSYQM can be obtained from it by applying the special unitary transformation [6]

$$\psi(x) \rightarrow \tilde{\psi}(x) = U\psi(x) = \rho_+\psi(x) - \rho_-\psi(-x), \quad \rho_{\pm} = \frac{1}{2}(1 \pm \sigma_1), \quad (8)$$

with subsequent reduction to one of the eigenspaces of σ_3 . Note that (8) is the *non-local* (bi-local) transformation. Nonlocality of bosonized SUSYQM is also hidden in the nature of the grading operator R : in coordinate representation it can be represented in the form $R = \sin(\pi H_0)$, $H_0 = \frac{1}{2}(x^2 - d^2/dx^2)$ [5].

Though the described relation in the direction (5)→(6), (7) is simple, the inverse correspondence is not clear: if we are given some bosonized supersymmetric system (e.g., with $W_-(x) = -x$), what is the system of the form (5) to be isospectral to it?

The bosonized supersymmetry can be treated as a supersymmetry of the system of two identical fermions living on a line: it is realized in their rest frame system in the $j_3 = 0$ eigensubspace of the total vector spin operator \mathbf{J} , $J_i = \frac{1}{2}(\sigma_i \otimes 1 + 1 \otimes \sigma_i)$, $i = 1, 2, 3$ [6].

Bosonized version of $N = 1$ SUSYQM can directly be extended for the two-dimensional space [6]:

$$Q_1 = \frac{1}{\sqrt{2}}(\pi_1 + i\pi_2 R), \quad Q_2 = iRQ_1, \quad H = \frac{1}{2}(\pi_a \pi_a - RB(x)),$$

where $\pi_a = -i\partial/\partial x^a - A_a(x)$, $a = 1, 2$, $A_a(x)$ is an odd ‘electromagnetic potential’, $A_a(-x) = -A_a(x)$, B is a ‘magnetic’ field, $B(x) = \partial_1 A_2(x) - \partial_2 A_1(x)$, and R is given by $R = R_1 R_2$, where R_1, R_2 are reflection operators with respect to x^1 and x^2 : $\{R_1, x^1\} = \{R_2, x^2\} = 0$, $[R_1, x^2] = [R_2, x^1] = 0$, $R_1^2 = R_2^2 = 1$, $[R_1, R_2] = 0$. In this case the operator R has also the sense of the operator of space rotation by the angle π , $R = \exp(-i\pi L)$, where $L = i(x^2 \partial_1 - x^1 \partial_2)$ is the operator of orbital angular momentum.

There is another interesting application of the bi-local unitary transformation (8): being applied to the Dirac field in 1+1 dimensions, it turns out to be analogous to the Foldy-Wouthuysen (FW) transformation. Indeed, let us consider the unitary transformation $\Psi(t, x) \rightarrow \tilde{\Psi}(t, x) = U\Psi(t, x)$, $U = \varrho_+ - \varrho_- R$, $\varrho_{\pm} = \frac{1}{2}(1 \pm \gamma_5)$, $\gamma_5 = \gamma^0 \gamma^1$, applied to the Dirac field satisfying the equation

$$[i(\gamma^0 \partial_t + \gamma^1 \partial_x) - m]\Psi(t, x) = 0.$$

Here we assume that R is the space reflection (parity) operator, $Rt = tR$, $Rx = -xR$, $R^2 = 1$. In representation $\gamma^0 = \sigma_3$, $\gamma^1 = i\sigma_2$, the unitary operator U takes exactly the form of the operator defining the transformation (8). The transformed Dirac field obeys the equation

$$[iR(\partial_t + \partial_x) + m\sigma_3]\tilde{\Psi}(t, x) = 0,$$

or, equivalently, the equation $i\partial_t \tilde{\Psi} = \tilde{H} \tilde{\Psi}$ with the Hamiltonian $\tilde{H} = -i\partial_x - R\sigma_3 m$, which has a diagonal matrix form. Though \tilde{H} has not the FW square root form, nevertheless, the relation $\tilde{H}^2 = -\partial_x^2 + m^2$ is valid due to the dependence of the Hamiltonian on the reflection operator R . If, like in the SUSYQM case, we further realize a reduction of the system to the eigensubspaces of σ_3 , we arrive finally at the linear differential equation with reflection

$$[iR(\partial_t + \partial_x) + \epsilon m]\psi_{\epsilon} = 0, \quad \epsilon = +, -, \quad (9)$$

being a square root of the Klein-Gordon equation for the one-component field ψ_{ϵ} since the equation $(-\partial_t^2 + \partial_x^2 - m^2)\psi_{\epsilon} = 0$ appears as the consequence of (9). The reduction, however, destroys the Poincaré invariance: the theory for one-component field ψ_{ϵ} is not invariant under either Lorentz transformations or space translations.

Equation (9) admits an extension for arbitrary dimension by introducing the set of operators R_i , $R_i x_i = -x_i R_i$ (no summation), $R_i x_j = x_j R_i$, $i \neq j$, $R_i t = t R_i$. E.g., in the (2+1)-dimensional case the nontrivial operator term from Eq. (9) is substituted for $\Delta = iR_2[R_1(\partial_0 + \partial_1) + \partial_2]$, $\Delta^2 = -\partial_0^2 + \partial_1^2 + \partial_2^2$. The corresponding d -dimensional one-component square root of the Klein-Gordon equation admits introducing the interaction with external gauge fields (for the details see ref. [6]).

It seems that the bosonized SUSYQM can be related to the (1+1)-dimensional integrable systems on the half-line. Besides, possibly, there is a relation between one-component linear differential equation (9) and the theory of massless boson and fermion fields on the half-line (or relation between d -dimensional generalization of Eq. (9) and the theory of corresponding massless fields living in the subspace $x_i \geq 0$). The latter possibility is due to the appearance of mass parameter in the boundary conditions which also destroy space translation and Lorentz symmetries [7].

Let us return to the bosonized form of SUSYQM (6), (7). In the case of the choice of the superpotential $W(x) = -\frac{\nu}{2x}$, $\nu \in \mathbb{R}$, the hermitian linear combination $i(Q_- - Q_+)$ of supercharges from (6) takes the form of the Yang-Dunkl operator $D_\nu = -i(\frac{d}{dx} - \frac{\nu}{2x}R)$, related to the 2-particle Calogero model, where R plays the role of exchange operator [6]. With this operator, one can construct the analogs of bosonic operators b^\pm ,

$$a^\pm = \frac{1}{\sqrt{2}}(x \mp iD_\nu). \quad (10)$$

Together with the reflection operator R they form the R -deformed Heisenberg algebra (RDHA) [8]–[11],

$$[a^-, a^+] = 1 + \nu R, \quad \{R, a^\pm\} = 0, \quad R^2 = 1. \quad (11)$$

This algebra possesses infinite-dimensional unitary representations for the values of deformation parameter $\nu > -1$, and at the integer values $\nu = p - 1$, $p = 1, 2, \dots$, it defines the parabosons of order p [9, 11]. At $\nu = -(2p + 1)$, this algebra defines (deformed) parafermions of order p [9, 11] (see below).

The Hamiltonian of bosonized supersymmetric systems with the superpotentials $W(x) = \epsilon x - \frac{\nu}{2x}$, $\epsilon = +, -$, can be represented in terms of generators of algebra (11),

$$H_\epsilon = \frac{1}{2}\{a^+, a^-\} - \frac{1}{2}\epsilon R[a^-, a^+]. \quad (12)$$

With the help of the number operator

$$N = \frac{1}{2}\{a^+, a^-\} - \frac{1}{2}(1 + \nu), \quad (13)$$

$[N, a^\pm] = \pm a^\pm$, and the relation $R = (-1)^N$, one can easily find that in the case $\nu > -1$, $\epsilon = +$, the Hamiltonian (12) gives the family of supersymmetric systems isospectral to the superoscillator system (1) (more exactly, to the system with the Hamiltonian $\tilde{H} = 2H$, where H is given by (1)). On the other hand, the case $\nu > -1$, $\epsilon = -$, describes the family of systems being in the phase of spontaneously broken supersymmetry, for which the two lowest states possess positive energy, whose value is governed by the deformation parameter ν : $E_0 = 1 + \nu > 0$. When $\nu = 1$, RDHA (11) describes parabosons of order $p = 2$ ($a^- a^+ |0\rangle = 2|0\rangle$, $a^- |0\rangle = 0$), and the Hamiltonian (12) takes a normal, $H_+ = a^+ a^-$, or antinormal, $H_- = a^- a^+$, form.

What will happen with supersymmetry if we take the Hamiltonian in the same normal or antinormal form in terms of a^\pm operators corresponding to the values of ν different from 1? Using Eqs. (11), (13), we find that for $\nu = 2k+1$, i.e. for parabosons of order $p = 2k+2$, the Hamiltonian $H_+ = a^+ a^-$ reveals a supersymmetric structure. The total spectrum is given by the series $E = 0, \dots, 2(k-1), 2k, 2k+2, 2k+2, 2k+4, 2k+4, \dots$, i.e., the peculiarity of such systems consists in the presence of $k+1$ lower lying singlet states (with the lowest state $|0\rangle$ of zero energy) instead of one singlet state of zero energy which we have in the case of usual supersymmetry. Therefore, such supersymmetric systems (for $k = 1, 2, \dots$) cannot be described by the bosonized version of supersymmetric quantum mechanics. To clarify their nature, it is sufficient to find the integrals of motion Q_\pm , $[H_+, Q_\pm] = 0$, being nilpotent supercharges, $Q_\pm^2 = 0$, which transform mutually the supersymmetric doublet states and annihilate singlet states. The sought for operators have the form

$$Q_+ = (a^+)^{2k+1} \Pi_-, \quad Q_- = (a^-)^{2k+1} \Pi_+, \quad (14)$$

and satisfy the relation

$$\{Q_+, Q_-\} = \mathcal{P}_{2k+1}(H_+), \quad (15)$$

where $\mathcal{P}_{2k+1}(H_+)$ is the polynomial in H_+ of order $p-1 = \nu = 2k+1$: $\mathcal{P}_{2k+1}(H_+) = (H_+ - 2k)(H_+ - 2k+2) \dots (H_+ + 2k-2)(H_+ + 2k)$.

For parabosons of order $p = 2k+2$, ($\nu = 2k+1$), $k = 1, \dots$, the systems given by the shifted antinormally ordered quadratic Hamiltonian $H_a = H_- - 2$ are also characterized by the nonlinear supersymmetry with supercharge operators having the form (14) with projectors Π_+ and Π_- to be changed in their places. In this case the polynomial appearing in the anticommutator of supercharges is $\mathcal{P}_{2k+1}(H_a) = (H_a - 2k+2) \dots (H_a + 2k)(H_a + 2k+2)$. The parabosonic system with the Hamiltonian H_a is characterized by the presence

of k singlet states with the state $|1\rangle = a^+|0\rangle$ playing the role of the ground state of zero energy. This state is odd, $R|1\rangle = -|1\rangle$, unlike the case of bosonized supersymmetric quantum mechanics (6), (7), where the ground state of zero energy (if exists, i.e. in the case of exact supersymmetry) is always even [6]. This explains, in particular, why the system described by the Hamiltonian H_a in the case of parabosons of order $p = 4$ has exactly the same spectrum as the parabosonic system of order $p = 2$ with the Hamiltonian H_+ (i.e. of the form of the spectrum of the superoscillator (1)), but is characterized by the nonlinear supersymmetry with the polynomial $\mathcal{P}_3(H_a)$ of order 3.

If we remember the coordinate realization (10), the Hamiltonians H_+ and H_a can be represented in the form

$$\mathcal{H}_\epsilon = \frac{1}{2} \left(-\frac{d^2}{dx^2} + x^2 + \frac{\nu^2}{4x^2} - 2 + \epsilon + \nu \left(\frac{1}{2x^2} - \epsilon \right) R \right), \quad (16)$$

where $\epsilon = +1$ corresponds to H_+ and $\epsilon = -1$ corresponds to H_a . The system given by the Hamiltonian (16) can be treated as a 2-particle Calogero-like model with exchange interaction, where x has a sense of relative coordinate and R has to be understood as an exchange operator. Therefore, at $\nu = 2k + 1$ the class of Calogero-like systems (16) possesses hidden supersymmetry with the supercharges given by

$$Q_+ = (Q_-)^\dagger = \frac{1}{2^{3(k+\frac{1}{2})}} \left(\left(-\frac{d}{dx} + x + \epsilon \frac{\nu}{2x} \right) (1 - \epsilon R) \right)^{2k+1},$$

and characterized by the superalgebra of the form (15). In correspondence with the discussion above, this supersymmetry can also be treated as a supersymmetry realized in the sector $j_3 = 0$ of the system of two identical fermions on a line.

Nonlinear (polynomial) supersymmetries appearing in the systems of parabosons of even order can be obtained via appropriate modification of classical analog for Witten's supersymmetric quantum mechanics. Indeed, the system described by the Lagrangian [12]

$$L = \frac{1}{2} (\dot{x}^2 - W^2(x) + ikW'(x)\epsilon_{ab}\theta_a\theta_b + i\theta_a\theta_a),$$

with θ_a , $a = 1, 2$, real Grassmann variables, and $\epsilon_{ab} = -\epsilon_{ba}$, $\epsilon_{12} = 1$, is characterized by the two odd integrals of motion,

$$Q_k^\pm = (B^\pm)^k \theta^\mp,$$

where $\theta^\pm = \frac{1}{\sqrt{2}}(\theta_1 \pm i\theta_2)$, and $B^\pm = \frac{1}{\sqrt{2}}(W(x) \mp ip)$ are the classical analogs of f^\pm and of operators (4) with p being a momentum canonically conjugate to x . Together with the Hamiltonian

$$H_k = \frac{1}{2}(p^2 + W^2 - ikW'\epsilon_{ab}\theta_a\theta_b),$$

they satisfy the Poisson bracket relation

$$\{Q_k^+, Q_k^-\}_{PB} = -i(H_k)^k, \quad (17)$$

which, unlike the usual case $k = 1$, is nonlinear for $k = 2, \dots$. In the case of $W = x$, the quantization of the system modifies the form of the algebra due to the quantum corrections and the quantum analog of (17) takes the form of the polynomial supersymmetry analogous to the supersymmetry in pure parabosonic systems with the Hamiltonians quadratic in creation-annihilation operators. However, in the case of the superpotential different from the linear one, generally the quantum anomalies arise which destroy supersymmetry algebra [12]. Note that the polynomial supersymmetry appeared also in other contexts in [13, 14].

The RDHA (11) related to the single-mode paraboson systems can be given in equivalent form by specifying it via the relations [12]

$$a^+a^- = \mathcal{F}(N), \quad a^-a^+ = \mathcal{F}(N+1), \quad [N, a^\pm] = \pm a^\pm, \quad (18)$$

where $\mathcal{F}(N) \equiv N(-1)^N + \nu \sin^2 \frac{\pi N}{2}$ is the so called structure function, satisfying for $\nu > -1$ the relations $\mathcal{F}(0) = 0$, $\mathcal{F}(n) > 0$, $n = 1, 2, \dots$. Then the hidden supersymmetries of parabosonic systems are encoded in the symmetry relations $\mathcal{F}(2n+1) = \mathcal{F}(2n+\nu+1)$ taking place for $\nu = 2k+1$, $k = 0, 1, \dots$. On the other hand, in the case $\nu = -(p+1)$, $p = 2, 4, \dots$, the structure function is characterized by the relation $\mathcal{F}(p+1) = 0$, which signals on existence of $(p+1)$ -dimensional irreducible representations of the algebra (11), in which the relations $(a^\pm)^{p+1} = 0$ are valid. However, in this case the function $\mathcal{F}(N)$ is not positive-definite, and the corresponding finite-dimensional representations are not unitary. Defining new creation-annihilation operators $f^- = a^-R$, $f^+ = a^+$, we arrive at the following parafermionic type algebra of the even order [9]:

$$\{f^-, f^+\} = p+1-R, \quad (f^\pm)^{p+1} = 0, \quad \{R, f^\pm\} = 0,$$

with $p = 2k$. Due to the presence of the reflection operator, the algebra has the natural internal \mathbb{Z}_2 structure and is characterized by the positive-definite structure function

$$\mathcal{F}(N) = N(-1)^N + (p+1) \sin^2 \frac{\pi N}{2} \quad (19)$$

possessing the symmetry

$$\mathcal{F}(n) = \mathcal{F}(p+1-n), \quad n = 0, \dots, p+1. \quad (20)$$

The parafermionic type systems of order p can be described by the relations of the same form (18) with structure functions obeying the relations $\mathcal{F}(0) = \mathcal{F}(p+1) = 0$, $\mathcal{F}(n) > 0$, $n = 1, \dots, p$ [15, 16]. Besides the described parafermions with internal \mathbb{Z}_2 structure, the symmetry relation (20) is satisfied by the structure functions describing other parafermionic systems including the ordinary parafermions ($\mathcal{F}(N) = N(p+1-N)$), finite-dimensional q -deformed oscillator ($\mathcal{F}(N) \equiv \mathcal{F}_q(N) = \sin \frac{\pi N}{p+1} / \sin \frac{\pi}{p+1}$, $q = \exp i \frac{\pi}{p+1}$) and the q -deformed parafermionic oscillator ($\mathcal{F}(N) = \mathcal{F}_q^2(N)$). Due to the symmetry (20), all the listed parafermionic systems with the quadratic hamiltonians $H_n = f^+ f^-$, $H_a = f^- f^+$ and $H_s = f^+ f^- + f^- f^+$ reveal a typical doubling of levels and, as a consequence, describe the supersymmetric systems. The details on the hidden supersymmetry in such parafermionic systems can be found in ref. [17], and here we restrict ourselves only by several general comments. In parafermionic case, due to the symmetry relation (20), the systems with H_n and H_a are equivalent and one of them can be obtained from another via the formal substitutions $f^+ \rightarrow f^-$, $f^- \rightarrow f^+$, $n \rightarrow p-n$. In the case of even p , the spectrum of the systems given by H_n has one singlet state ($|0\rangle$) of zero energy, whereas for odd p it contains additional singlet level ($|(p+1)/2\rangle$) of non-zero energy. As a consequence, in the first case the corresponding anticommutator of supercharges is linear in Hamiltonian, whereas in the second case it is given by the quadratic in H_n polynomial. The Hamiltonian H_s describes supersymmetric systems in the phase of spontaneously broken supersymmetry (there are no singlet states of zero energy), in which all the states are supersymmetric doublets for odd p and there is one singlet state ($|p/2\rangle$) of positive energy for even p . In both cases the supersymmetry is linear. In parafermionic systems with quadratic Hamiltonians more complicated cases characterized by the anticommutator of supercharges being the polynomial of order higher than 2 can also be obtained. E.g., in the case of ordinary parafermions ($\mathcal{F}(N) = N(p+1-N)$) of order $p \geq |k| + 1$, $|k| \geq 2$, the systems given by the Hamiltonian

$$H_k = \{f^+, f^-\} + k[f^+, f^-] + p(|k| - 1), \quad k \in \mathbb{Z},$$

are characterized by the polynomial supersymmetry whose order is $|k|$ ($|k| + 1$) when the difference $p - k$ is odd (even).

We conclude that the hidden supersymmetries of pure parabosonic and parafermionic systems can be explained on the general ground by the corresponding symmetry properties of the structure functions. At the same

time, the possibility of realizing supersymmetry in pure parafermionic case is not so surprising: parafermions generalize fermions with their basic relations $\{f^+, f^-\} = 1$, $(f^\pm)^2 = 0$, which can be treated as specifying a trivial supersymmetric system with the supercharges $Q_\pm = f^\pm$ and the Hamiltonian $H = 1$. On the other hand, the existence of hidden supersymmetries in pure parabosonic systems with quadratic Hamiltonians seems to be rather peculiar. It can be given the following interpretation. One can represent the corresponding supercharge operators (14) in terms of only creation-annihilation operators by using the relation $R = (-1)^N$ and Eq. (13). In this way we write the projectors Π_\pm in the equivalent form [12]

$$\Pi_+ = \cos^2 \mathcal{N}, \quad \Pi_- = \sin^2 \mathcal{N}, \quad \mathcal{N} = \frac{\pi}{4} \{a^+, a^-\}.$$

Therefore, though the corresponding Hamiltonians $H_+ = a^+ a^-$ and $H_a = a^- a^+ - 2$ are quadratic in operators a^\pm , the price we are paying to have supersymmetry in pure parabosonic systems is the specific structure of the supercharges: they are the infinite series in a^\pm of the special form which gives rise to their nilpotency and to the polynomial form for the anticommutator (15). However, it is not clear what are the corresponding supersymmetry transformations for such pure parabosonic systems and for the related minimally bosonized supersymmetric quantum mechanics. Probably, the construction of classical models for them could help to answer this question.

Acknowledgements The work was supported in part by the grant 1980619 from FONDECYT (Chile) and by DICYT (USACH).

References

- [1] H. Nicolai, *Supersymmetry and spin systems*, J. Phys. A: Math. Gen. **9** (1976) 1497.
- [2] E. Witten, *Dynamical breaking of supersymmetry*, Nucl. Phys. **B188** (1981) 513; *Constraints on supersymmetry breaking*, **B202** (1982) 253.
- [3] L. E. Gendenshtein and I. V. Krive, *Supersymmetry in quantum mechanics*, Sov. Phys. Usp. **28** (1985) 645; F. Cooper, A. Khare and U. Sukhatme, *Supersymmetry and quantum mechanics*, Phys. Rep. **251** (1995) 267, hep-th/9405029.

- [4] M. S. Plyushchay, *Deformed Heisenberg algebra, fractional spin fields and supersymmetry without fermions*, Ann. Phys. **245** (1996) 339, hep-th/9601116.
- [5] M. S. Plyushchay, *Minimal bosonization of supersymmetry*, Mod. Phys. Lett. **A11** (1996) 397, hep-th/9601141.
- [6] J. Gamboa, M. Plyushchay and J. Zanelli, *Three aspects of bosonized supersymmetry and linear differential field equation with reflection*, Nucl. Phys. **B543** (1999) 447, hep-th/9808062.
- [7] A. Liguori and M. Mintchev, *Quantum field theory, bosonization and duality on the half line*, Nucl. Phys. **B522** (1998) 345, hep-th/9710092.
- [8] M. S. Plyushchay, *R-deformed Heisenberg algebra*, Mod. Phys. Lett. **A11** (1996) 2953, hep-th/9701065.
- [9] M. S. Plyushchay, *Deformed Heisenberg algebra with reflection*, Nucl. Phys. **B491** (1997) 619, hep-th/9701091.
- [10] M. S. Plyushchay, *R-deformed Heisenberg algebra, anyons and $D=(2+1)$ supersymmetry*, Mod. Phys. Lett. **A12** (1997) 1153, hep-th/9705034.
- [11] M. S. Plyushchay, *Universality of the R-deformed Heisenberg algebra*, Lecture Notes in Physics, vol. **509**, p. 289 (Springer-Verlag, Berlin, 1998), hep-th/9705043.
- [12] M. Plyushchay, *Hidden nonlinear supersymmetries in pure parabosonic systems*, hep-th/9903130, to appear in Int. J. Mod. Phys. **A**.
- [13] A. Andrianov, M. Ioffe and V. Spiridonov, *Higher-derivative supersymmetry and the Witten index*, Phys. Lett. **A174** (1993) 273, hep-th/9303005; A. Andrianov, F. Cannata, M. Ioffe and D. Nishniadze, *Systems with higher-order shape invariance: spectral and algebraic properties*, quant-ph/9902057 and references therein.
- [14] A. P. Isaev and R. P. Malik, *Deformed traces and covariant quantum algebras for quantum groups $GL_{qp}(2)$ and $GL_{qp}(1/1)$* , Phys. Lett. **B280** (1992) 219.
- [15] D. Bonatsos, C. Dascaloyannis and P. Kolokotronis, *Generalized deformed oscillators and algebras*, hep-th/9512083 and references therein.

- [16] C. Quesne, *Generalized parafermions, nonlinear deformations of $so(3)$ and exactly solvable potentials*, Phys. Lett. **A193** (1994) 245.
- [17] S. Klishevich and M. Plyushchay, *Supersymmetry of parafermions*, hep-th/9905149, to appear in Mod. Phys. Lett. **A**.